STABILITY OF CONVECTIVE MOTION IN A TWO-DIMENSIONAL VERTICAL FLUID LAYER WITH PERMEABLE BOUNDARIES

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The stability of the convective motion of a viscous incompressible fluid in a channel between permeable vertical planes heated to different temperatures is considered under the assumption of homogeneous transverse air blasting. Stability boundaries for different values of the Prandtl number Pr and Peclet number Pe that characterize the intensity of transverse motion are numerically determined. The results demonstrate that transverse blasting substantially influences both the hydrodynamic instability mechanism and instability due to the growth of thermal waves in the flow.

The stability of steady convective motion between vertical parallel planes heated to different temperatures has been investigated in detail [1-7]. The existence of monotonic-type instability of a hydrodynamical nature and oscillatory type due to the growth of thermal waves in the flow was demonstrated. It was assumed that the channel walls were impermeable to the substance and that neither suction nor draining of the fluid from the boundaries occurred, though injection and drawing off of fluid through permeable boundaries can exert a substantial influence on the stability of the resulting steady motion and may serve as one method of controlling hydrodynamical and convective instability. It is well known that transverse motion leads to a significant increase in the stability of a laminar boundary layer [8, 9] and of plane Poiseuille flow [10, 11]. It was proved [12] that transverse draining increases the critical Rayleigh number that determines the appearance of convection in a horizontal layer heated from below.

§1. Let us consider a plane vertical layer of a viscous incompressible fluid bounded by the infinite planes $x=\pm h$, heated to the different temperatures $\pm\theta$. Suppose one-dimensional injection of a fluid at the rate v_0 occurs on the surface $x=-h$ and that one-dimensional draining occurs on the surface $x=h$ at the same rate. Thus, the resulting steady motion is the superposition of a one-dimensional transverse flow on plane-parallel convective flow,

$$v_x=v_0,\ v_y=0;\ v_z=u_0(x),$$

where $v_0=\text{const}$.

The steady convection equations have the form

$$\frac{\mathrm{Pe}}{\mathrm{Pr}}u_0'-u_0''-T_0=-\frac{\partial p_0}{\partial z}=c, \tag{1.1}$$
$$\mathrm{Pe}T_0'=T_0''\quad\left(\mathrm{Pe}=\frac{v_0h}{\chi},\ \mathrm{Pr}=\frac{\nu}{\chi}\right),$$

where u_0, T_0, and p_0 are vertical velocity, absolute temperature, and pressure, respectively; P_e is the Peclet number that characterizes the intensity of transverse motion, Pr is the Prandtl number, ν is the kinematic viscosity coefficient, χ is the thermal-diffusivity coefficient, and c is the separation constant of the variables. We introduce h, h^2/ν, θ, $g\beta\theta h^2/\nu$, and $\rho g\beta\theta h$, where g is the acceleration of gravity, ρ is density, and β is the thermal-expansion coefficient, as the units of distance, time, temperature, velocity, and pressure.

A temperature is specified on the channel boundaries and the vertical component of velocity vanishes,

$$T_0(-1)=-1,\ T_0(1)=1,\ u_0(\pm1)=0. \tag{1.2}$$

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We add one further condition to the above. That is, fluid flow rate through a transverse channel cross section is zero,

$$\int_{-1}^{+1} u_0(x)\,dx = 0. \tag{1.3}$$

From Eqs. (1.1), taking into account Eqs. (1.2) and (1.3), we find that the temperature distribution and distribution of the vertical velocity component of the resulting motion are given by

$$T_0(x) = \frac{1}{\mathrm{shPe}}\left(e^{\mathrm{Pe}x} - \mathrm{chPe}\right); \tag{1.4}$$

$$u_0(x) = c_0\left(e^{\mathrm{Pe}x} - x\,\mathrm{shPe} - \mathrm{chPe}\right) + c_1\left(e^{\frac{\mathrm{Pe}}{\mathrm{Pr}}x} - x\,\mathrm{sh}\frac{\mathrm{Pe}}{\mathrm{Pr}} - \mathrm{ch}\frac{\mathrm{Pe}}{\mathrm{Pr}}\right), \text{ where}$$

$$c_0 = \frac{\mathrm{Pr}}{\mathrm{Pe}^2(1-\mathrm{Pr})\,\mathrm{sh\,Pe}};$$

$$c_1 = -\frac{c_0\left(\mathrm{chPe} - \frac{1}{\mathrm{Pe}}\,\mathrm{shPe}\right)}{\mathrm{ch}\frac{\mathrm{Pe}}{\mathrm{Pr}} - \frac{\mathrm{Pr}}{\mathrm{Pe}}\,\mathrm{Sh}\frac{\mathrm{Pe}}{\mathrm{Pr}}}.$$

In passing to limits we obtain from Eqs. (1.4) a linear temperature distribution $T_0 = x$ and cubic velocity profile $u_0 = (x/6)/(1-x^2)$, which holds for a layer with impermeable boundaries in the absence of transverse fluid flow ($\mathrm{Pe}=0$). The presence of a transverse velocity component ($\mathrm{Pe} \neq 0$) leads to distortion of the P_0 profile, which ceases being a linear function of the transverse coordinate. A thermal boundary layer forms near one of the boundaries at high Pe. Transverse motion leads also to distortion of the stationary distribution of the vertical velocity component. Two mechanisms that deform u_0 exist here:

a) a convective mechanism associated with the distortion of the stationary temperature distribution;

b) a hydrodynamical mechanism associated with the interaction of the transverse flow with the convective plane-parallel flow and described by the nonlinear terms of the Navier–Stokes equation.

Figure 1 depicts stationary temperature (T_0) and velocity (u_0) distributions for $\mathrm{Pr}=2$ and $\mathrm{Pe}=3$. Values of T_0 and u_0 in the case of impermeable boundaries are indicated by a prime for comparison. Transverse motion leads to a decrease in flow rate and to asymmetry of the profile of the vertical velocity component.

§ 2. We will write equations for small disturbances of the stationary temperature and velocity distributions in order to study the stability of the resulting nonparallel fluid motion. An analysis demonstrates that (as in the case of flow) a crisis in the flow between impermeable boundaries [13] is associated with the development of two-dimensional disturbances, whose description requires that we introduce the stream function $\psi(x, z, t)$, connected to the velocity components by the equations

$$v_x = \frac{\partial\psi}{\partial z};\quad v_y = 0;\quad v_z = -\frac{\partial\psi}{\partial x}.$$

The system of equations for disturbances in the stream and temperature functions has the form

$$\begin{gathered}\frac{\partial}{\partial t}\Delta\psi + \frac{\mathrm{Pe}}{\mathrm{Pr}}\frac{\partial}{\partial x}\Delta\psi + \mathrm{Gr}\left(u_0\frac{\partial}{\partial z}\Delta\psi - u_0''\frac{\partial\psi}{\partial z}\right) = \Delta\Delta\psi - \frac{\partial T}{\partial x};\\ \frac{\partial T}{\partial t} + \frac{\mathrm{Pe}}{\mathrm{Pr}}\frac{\partial T}{\partial x} + \mathrm{Gr}\left(u_0\frac{\partial T}{\partial z} + T_0'\frac{\partial\psi}{\partial z}\right) = \frac{1}{\mathrm{Pr}}\Delta T,\end{gathered} \tag{2.1}$$

where

$$\mathrm{Gr} = \frac{g\beta\theta h^3}{\nu^2} \text{ is the Grashof number; } \Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}.$$

We introduce disturbances of the form

$$\begin{gathered}\psi(x, z, t) = \varphi(x)e^{-\lambda t + ikz};\\ T(x, z, t) = \tau(x)e^{-\lambda t + ikz},\end{gathered} \tag{2.2}$$

where $\varphi(x)$ and $\tau(x)$ are disturbance amplitudes, k is a real wave number, and $\lambda = \lambda_r + \lambda_i$ is the complex disturbance decrement. The stability boundary is determined by the condition $\lambda_r = 0$; the imaginary part λ_i of the decrement determines the phase velocity of the disturbance.

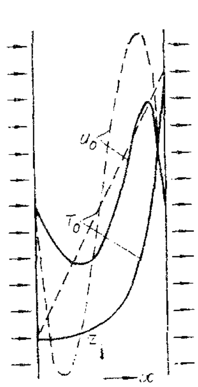

Fig. 1

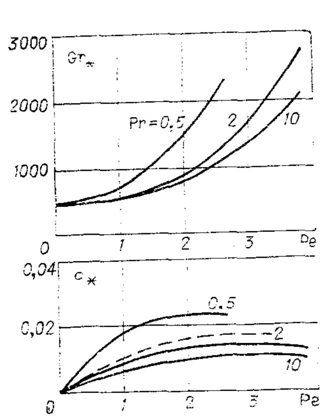

Fig. 2

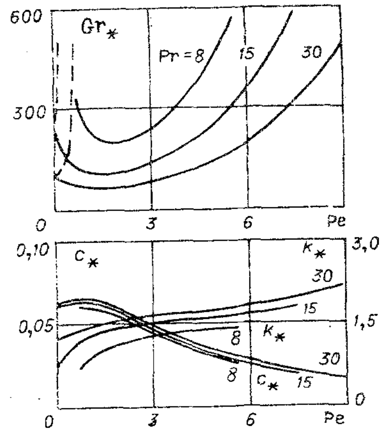

Fig. 3

We substitute Eq. (2.2) in (2.1), obtaining a system of equations for the disturbance amplitudes,

$$ik\,\mathrm{Gr}\left[u_0(\varphi''-k^2\varphi)-u_0''\varphi\right]-\lambda(\varphi''-k^2\varphi)=(\varphi^{\mathrm{IV}}-2k^2\varphi''+k^4\varphi)-\frac{\mathrm{Pe}}{\mathrm{Pr}}(\varphi'''-k^2\varphi')-\tau'; \tag{2.3}$$

$$ik\,\mathrm{GrPr}\left(u_0\tau+T_0'\varphi\right)-\lambda\mathrm{Pr}\tau=(\tau''-k^2\tau)-\mathrm{Pe}\tau'.$$

Conditions under which the temperature and velocity disturbances on the wall vanish lead to the homogeneous boundary conditions

$$\tau(\pm1)=\varphi(\pm1)=\varphi'(\pm1)=0. \tag{2.4}$$

Numerical integration was used to solve the characteristic spectral problem (2.3), (2.4). The system of equations (2.3) for the complex amplitudes φ and τ reduces to a system of 12 real first-order differential equations for the real and imaginary parts of the functions φ, φ', φ'', φ''', and τ'. Three linearly independent partial solutions of this system were constructed using the Runge–Kutta–Merson method, these solutions satisfying the conditions (2.4) at the point $x=-1$; "0–1"-type conditions for the higher derivatives were also constructed. An orthogonalization procedure [14] was used to retain the linear independence of the solutions over the entire range of integration. The boundary conditions at the right end $x=1$ of the range of integration lead to characteristic equations that determine the real and imaginary parts of the decrement λ.

§3. In order to present the results of the calculation we will enumerate the basic results of the study of the stability of convective flow between impermeable boundaries.

Convective flow in a vertical layer with impermeable boundaries reveals two types of instability as a function of the value of Pr. When Pr < 12, the flow and heat transfer crisis is hydrodynamical and associated with the instability of the interface between opposing convective flows. This crisis is due to an increase in the so-called "standing" disturbances, which lead to the formation of a vertically periodic chain of immobile eddies at the boundary of the flows. When Pr > 12, instability is due to "travelling" disturbances in the form of thermal waves increasing in the flow. The phase velocity of these waves is commensurate with the velocity of the main flow, while there exist two waves propagating in the ascending and descending flows, respectively. These waves have phase velocities identical in magnitude and lead to the appearance of instability at identical Grashof numbers.

Such features involved in the appearance of instability are basically due to the asymmetry of the velocity and temperature profiles of the main fluid motion in the layer between impermeable boundaries. Standing disturbances with vanishing phase velocity are impossible as in the case of convective flow of a fluid with temperature-dependent viscosity. The hydrodynamical mechanism of the instability of the interface between opposing flows is now associated with disturbances that slowly drift along a vertical line upwards in the direction of the motion of more intensive flow. Calculations have demonstrated that their phase velocity is near in magnitude to the velocity $u=u_1-u_2$, where $u_1\equiv u^+_{max}$ and $u_2\equiv u^-_{max}$ are the maximal velocities in the ascending and descending flows.

Figure 2 depicts the dependence of the critical (minimal along a neutral curve) value of the Grashof number Gr_* for hydrodynamic-type disturbances and the corresponding dimensionless phase velocity

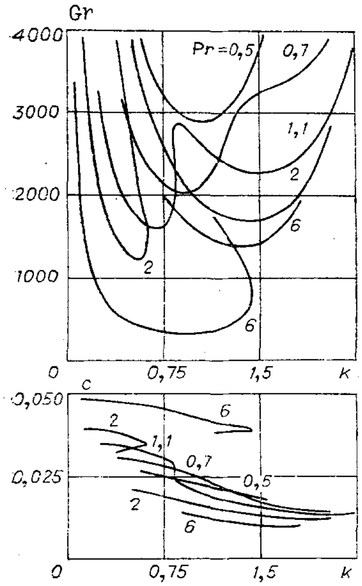

Fig. 4

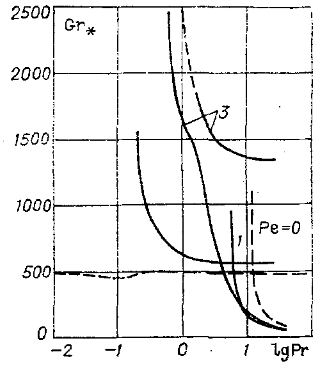

Fig. 5

$c_* = \lambda_{i*}/k_* Gr_*$ on the parameter Pe for the three values of the Prandtl numbers 0.5, 2, and 10. Transverse motion exerted the strongest stabilizing effect on the development of disturbances. The critical Grashof number for Pr = 0.5 was roughly three times greater than for flow with a cubic velocity profile in the case of a comparatively low transverse speed (Pr = 2). When Pe > 2.5, Gr_* increases in proportion to Pe^2 for all these values of the Prandtl number. The velocity of the critical disturbances increases with increasing Pe and reaches a maximal value for some Peclet number. A further growth in the transverse speed leads to a decrease in c_*. The dependence of velocity u on Pe for Pr = 2 is depicted in Fig. 2 by a broken line. The behavior of u(Pe) is analogous to that of c_* (Pe), while u is always greater in magnitude than c_*, but the difference between u and c_* decreases with decreasing Prandtl number.

The calculations demonstrated that the critical wave number k_* determining the wavelength of the most dangerous disturbances remains practically invariant throughout the hydrodynamical branch of the instability. A variation in the Pe and Pr parameters resulted in $k_* \approx 1.4$ in this range. Let us consider instability induced by a growth of thermal waves. The basic difference from the case of impermeable boundaries [4] is that no "combining" of the real levels with the generation of a pair of oscillating disturbances occurs in the decrement spectrum of λ(Gr). Asymmetry of the T_0 and u_0 profiles in the case of the homogeneous transverse velocity results in the fact that thermal waves propagating upwards and downwards cease being equal, have phase velocities differing in magnitude, and different instability boundaries correspond to them. The influence of homogeneous transverse motion on the stability boundary and the characteristics of the critical disturbances are shown in Fig. 3, where dependences are given for the critical Grashof numbers Gr, phase velocities c_*, and wave numbers k_* for thermal waves propagating in the positive direction of the z axis (solid curves). The dependence of Gr_* (Pe) for thermal waves with negative phase velocity is depicted by broken curves. The dependences Gr_* (Pe) for "positive" waves have a minimum at approximately Pe = 1.6 for Pr = 15 and 30 and when Pe = 1.9, for Pr = 8. Thus, flow becomes less stable at low transverse velocities than in the case of impermeable boundaries for thermal waves with positive phase velocity. Flow stability increases with strong fluid suctions and drains on the boundaries and Gr_* increases in proportion to Pe^2 when Pe > 5.5.

An increase in stability for all values of Pe is observed and the disturbances are "deflated" from the descending flow in the case of thermal waves with negative phase velocity.

The curves for c_* (Pe) demonstrate that the phase velocity of the critical disturbances somewhat increases with increasing transverse velocity. The critical phase velocity sharply increases in the range of values of Peclet numbers in which the destabilizing influence of the draining is replaced by a stabilizing influence. It becomes near in magnitude to the maximal velocity of the ascending flow. An increase in the critical wave number k_* with increasing Pe is observed for thermal waves with critical phase velocity in contrast to the hydrodynamical instability mode for thermal waves with positive phase velocity. That is, the wavelength of disturbances defining a crisis of the given flow decreases with increasing transverse velocity.

Results of the calculation demonstrated that weak transverse draining significantly decreases the limiting Pr_* at which such instability in the form of increasing thermal waves, appears. While $Pr_* = 11.4$ [4] in the case of impermeable boundaries (Pe = 0), we obtain $Pr_* = 5$ for Pe = 1 (we are bearing in mind a wave with positive phase velocity). As before, there exist two different neutral curves, one of which determines the instability region relative to hydrodynamical disturbances, and the other, relative to thermal waves for all Pr > 5.

Strong transverse draining leads to qualitatively new results in the region of low and medium values of the Prandtl number. Figure 4 depicts neutral curves on the k and Gr planes and phase velocities along neutral curves for several values of the Prandtl number. The Peclet number was assumed fixed and equal to 3. A new instability mode is obtained as a result of the continuous deformation of the single neutral curve as the Prandtl number increases, as in the case of convective flow induced by internal heat sources [16]. When Pr = 1.1, Gr(k) consists of two neutral curves that continuously pass into each other. The curve has two minima and we may correspondingly speak of two types of instability. A further increase in Pr leads to a division of the instability region and the formation of different neutral curves, one of which (short-wave) determines the instability region of hydrodynamical disturbances, while the other (long-wave) determines the "positive" thermal waves, associated with the most dangerous disturbance corresponding to an absolute minimum on the neutral curves.

Let us now present a summary of data on the stability boundary in the case of one-dimensional transverse draining. The dependence of Gr_* on the Prandtl number is depicted in Fig. 5 for given values of the Peclet number. The broken curves correspond to stability boundaries relative to hydrodynamical disturbances and thermal waves in the case of impermeable boundaries. Fluid draining and suction on the boundaries lead to flow destabilization relative to thermal waves for Pe = 1.3. On the other hand, flow stability relative to thermodynamical disturbances increases. A strong increase in stability on the hydrodynamical branch in the region of low Prandtl numbers is due to our selection of the characteristic parameters (Pe = const). The ratio Pe/Pr = Re (the Reynolds number is determined in terms of the transverse velocity) is high at low Pr. As has been previously proved [16], an increase in Re, even in the hydrodynamical formulation, leads to flow stabilization. The broken part of the curve for Pe = 3 refers to the short-wave minimum on the single neutral curve (cf. Fig. 5).

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